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Coulomb scattering in the Born approximation and the use of generalized functions

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Abstract

We discuss three approaches that have been used in obtaining the Born approximation for Coulomb scattering: The standard approach, making use of a convergence factor (“screening”), Oppenheimer’s approach using cylindrical (instead of spherical) coordinates, and finally Landau and Lifshitz’ approach. We simplify and clarify Oppenheimer’s calculation, which has been rarely used because of its complications, and make the Landau and Lifshitz method rigorous. Although the latter method does require some background from the theory of generalized functions, it is nevertheless a very instructive and important technique deserving more exposure to instructors and theoretical physicists in general.

I. INTRODUCTION

Our knowledge about elementary particles and their interactions is based mainly on scattering experiments, making scattering theory a very important subject. Quantum mechanics texts^{1–3} usually deal with scattering from a potential using the Born series expansion for the scattering amplitude and work out examples by evaluating only the lowest term of the series, the so-called Born approximation. In the present paper we shall be concerned with the calculation of the Born approximation for Coulomb scattering; however, for some applications of the Born approximation in strong interactions we refer the reader to Kozack.⁴

In a seminal 1911 paper on atomic structure, Rutherford⁵ derived the expression for the differential cross section of Coulomb scattering using only classical mechanics. (A derivation in a modern text may found in reference [6].) In fact, Rutherford introduced the concept of what we now call *the differential cross section* for potential scattering. As Weinberg² points out, Rutherford was lucky in that the Coulomb case is the only one for which the classical result agrees perfectly with the quantum result!

It was not until 1926 that Born^{7,8} used quantum mechanics to derive the series expansion for the scattering amplitude, whose lowest term is referred to as the Born approximation, f_B . Born also noted the fact that each term in this series diverges for the case of the Coulomb potential. Four months later, Wentzel⁹ evaluated f_B for Coulomb scattering in spherical coordinates by making use of a convergence factor. To our knowledge, *all quantum*

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mechanics texts perform the integration involved in calculating f_B using Wentzel's limiting procedure. However, in 1927 Oppenheimer¹⁰ derived the Born approximation for Coulomb scattering without the use of Wentzel's limiting procedure by performing the integrations in cylindrical coordinates. Unfortunately his calculation is unnecessarily complicated due to his choice of the orientation of the momentum transfer vector \vec{q} , and this is probably the reason it has disappeared from the literature.

In this paper, we greatly simplify and clarify Oppenheimer's calculation in the hope that this version will be a useful alternative to lecturers on this subject. Finally we present a calculation of f_B found in a lesser known Landau and Lifshitz text.¹¹ Unfortunately the presentation by Landau and Lifshitz involves a number of mathematically illegal steps. To put it another way, the derivation assumes, but does not mention, that the reader should be familiar with generalized function theory in order to justify the steps. We clarify the derivation and, in the Supplementary Material¹² give an exposition of the necessary generalized function theory that makes all the steps in reference [11] rigorous.

In Sec. II we review the standard calculation of the Born approximation for Coulomb scattering. In Sec. III we give the simplified version of Oppenheimer's approach. In Section IV we go over the Landau and Lifshitz derivation. The Supplementary Material¹² is required for a deeper understanding of Section IV and is devoted to an elementary presentation of all the necessary tools from the theory of generalized functions. Finally, in Sec. V we sum up our results. Appendix A deals with our choice of units and dimensions; Appendix B is a review of potential scattering; and in Appendix C we give some details of Oppenheimer's original calculation.

II. THE BORN APPROXIMATION I

We first outline the standard calculation of the scattering amplitude in the Born approximation. It is convenient to adopt Planck natural units so that, $c = \hbar = 1$; see Appendix A for more details. The momentum transfer vector \vec{q} , is given by (see Appendix B, Fig. 1)

$$\vec{q} = \vec{p}'_1 - \vec{p}_1, \quad (1)$$

where \vec{p}_1 and \vec{p}'_1 are, respectively, the initial and final momenta of one of the particles in the

center of mass frame. For elastic scattering, $|\vec{p}_1| = |\vec{p}_1'| = p$, and we have that,

$$q = 2p \sin \left(\frac{\theta}{2} \right), \quad (2)$$

where $\theta \in [0, \pi]$ and thus $q \in [0, 2p]$.

For a spherically symmetric potential energy, $V(r)$, the scattering amplitude in the Born approximation is given by Eq. (B31) in Appendix B 3,

$$f_B = -\frac{m}{2\pi} \int V(r) e^{-i\vec{q} \cdot \vec{r}} d^3r, \quad (3)$$

where $m = m_1 m_2 / (m_1 + m_2)$ is the reduced mass, so that for electron - proton scattering, $m \approx m_e$. When using spherical coordinates, (r, ϑ, ϕ) , to evaluate Eq. (3), it is important to bear in mind that the angle ϑ is not related to the scattering angle θ in Eq. (2). In performing the integrations, it is most convenient to choose the polar axis in the direction of the vector \vec{q} , so that the exponent $-i\vec{q} \cdot \vec{r} = -iqr \cos \vartheta$. Integrating over ϑ and ϕ , we obtain

$$f_B = -\frac{2m}{q} \int_0^\infty V(r) \sin(qr) r dr. \quad (4)$$

In the Gaussian system of units (see Appendix A), the Coulomb potential energy (or Coulomb potential for short) is given by

$$V(r) = \frac{\pm e^2}{r}, \quad (5)$$

where we have assumed for simplicity that the particles have charge $\pm e$. The signs, $(+, -)$, correspond to repulsion and attraction respectively. Equation (5) can easily be generalized for other charges.

The convergence of the integral in Eq. (4) in spherical coordinates requires that,¹³

$$\int_0^\infty r^2 |V(r)| dr < \infty. \quad (6)$$

Thus it is evident that $V(r)$ does not vanish sufficiently fast at large r to allow for a straightforward evaluation of the above integral.

The universally adopted procedure in the Coulomb case is to define f_B using the expression below (we use $:=$ for definitions)

$$f_B := \lim_{\lambda \rightarrow 0} \left(-\frac{2m}{q} \int_0^\infty V(\lambda, r) \sin(qr) r dr \right), \quad (7)$$

where $V(\lambda, r)$ is often referred to as the *screened* Coulomb potential,

$$V(\lambda, r) = \pm e^2 \left(\frac{e^{-\lambda r}}{r} \right). \quad (8)$$

In our units, the parameter λ has dimensions of inverse length (or mass), and so it is related to a cutoff length, r_c , by $\lambda = 1/r_c$. The exponential factor in Eq. (8) is a qualitative description for the screening of the nuclear charge of an atom by the electrons of the surrounding shells (regarded as a continuous charge density). A potential of the form (8) is also known as the Yukawa potential because it was proposed by Yukawa¹⁴ to describe the strong nucleon-nucleon interaction mediated by the exchange of a spinless boson of mass λ , (with g^2 instead of e^2).

Wentzel⁹, who carried out the original calculation of the Born approximation for the Coulomb potential, wrote: “So you cannot get by with the pure Coulomb field but at least qualitatively you have to take into account *the screening by the outer electron shells*” (our italics). Under these circumstances, the cutoff r_c would be on the order of an atomic radius. However he added that “...the special choice of the exponential function in [our Eq. (8)] is completely irrelevant for the end result; it only offers the advantage of ensuring the convergence of the method in the simplest possible way.” A lot of research on screening, both theoretical^{15,16} and experimental,^{17,18} has been carried out. Nevertheless it is important to recall that the Born approximation for Coulomb scattering is also valid for electron-electron scattering *where there is no screening due to a surrounding electron cloud as in atoms*. Here, one would have to introduce the idea of vacuum polarization of the virtual pairs and charge renormalization^{19–21} and make an intuitive argument that effectively, the target charge varies so that

$$e \rightarrow e \left(e^{-\lambda r} \right). \quad (9)$$

Oppenheimer's approach avoids these problems by obtaining f_B without the need of a convergence factor.

It is worth pointing out that higher-order terms in the Born series with the potential of Eq. (8) diverge in the limit $\lambda \rightarrow 0$,^{22,23} and require a special procedure in order to cancel the divergences.²⁴

We should add that Mott,²⁵ and independently Gordon,²⁶ calculated the *exact* Coulomb potential scattering amplitude by solving the Schrödinger equation. Although their approaches are very different, neither of them needed any convergence factor. Both calcula-

tions are rather demanding. Gordon's method, making use of parabolic coordinates, is the one worked out in detail as an exact solution, but only in more advanced quantum mechanics texts.^{1,22} Gordon's solution is another example where a judicious choice of the coordinate system is helpful.

In Wentzel's convergence factor approach, Eq. (7) is the Laplace transform of $\sin(qr)$. The integration is standard and is done by performing two integrations by parts, thus

$$f_B = \lim_{\lambda \rightarrow 0} \left(\mp \frac{2me^2}{q} \int_0^\infty e^{-\lambda r} \sin(qr) dr \right) \quad (10)$$

$$= \lim_{\lambda \rightarrow 0} \left(\mp \frac{2me^2}{q^2 + \lambda^2} \right) \quad (11)$$

$$= \mp \frac{2me^2}{q^2} \quad (12)$$

$$= \mp \frac{me^2}{2p^2 \sin^2\left(\frac{\theta}{2}\right)}, \quad (13)$$

where we have used Eq. (2).

III. THE BORN APPROXIMATION II

In this section, following Oppenheimer's approach, we begin with Eq. (3) and use cylindrical coordinates (z, ϱ, φ) . In contrast with his choice of the orientation of the momentum transfer \vec{q} (see Appendix C, Eq. (C11)), we choose the direction of the vector \vec{q} to be along the positive z -axis; thus the exponent in the integrand of Eq. (3) is $-i\vec{q} \cdot \vec{r} = -iqz$, which considerably simplifies Oppenheimer's original calculation. The Coulomb potential V , Eq. (5), is now given by

$$V(\varrho, z) = \pm \frac{e^2}{\sqrt{\varrho^2 + z^2}}. \quad (14)$$

The resulting integrals in this case are referred to as *iterated integrals*, i.e., the integrations over z and ϱ *have to be done in the order in Eq. 15 below*. (Oppenheimer does not mention this). The condition under which one is allowed to interchange the order of integration is given by Fubini's theorem,²⁷ and the integrals in Eq. (15) fail to satisfy this condition

because the integration of the absolute value of the integrand over z does not converge. Therefore we have that,

$$f_B := \mp \frac{me^2}{2\pi} \int_0^\infty \varrho d\varrho \int_{-\infty}^\infty \frac{e^{-iqz}}{\sqrt{\varrho^2 + z^2}} dz \int_0^{2\pi} d\varphi, \quad (15)$$

$$= \mp me^2 \int_0^\infty \varrho d\varrho \int_{-\infty}^\infty \frac{e^{-iqz}}{\sqrt{\varrho^2 + z^2}} dz, \quad (16)$$

$$= \mp 2me^2 \int_0^\infty \varrho K_0(q\varrho) d\varrho, \quad (17)$$

$$= \mp \frac{2me^2}{q^2}, \quad (18)$$

which is again Eq. (12). K_0 is a modified Bessel function of the second kind. The integrals involved in Eq. (16) and (17) may be found, for example, in reference [27], or may be obtained immediately using *Mathematica*²⁹ or any other symbolic computation software. In connection with the integral in Eq. (17), we remark that, although $\lim_{\varrho \rightarrow 0} [K_0(q\varrho)] \rightarrow \infty$, it turns out that $\lim_{\varrho \rightarrow 0} [\varrho K_0(q\varrho)] = 0$, so everything is well-behaved.

IV. THE BORN APPROXIMATION III

In this section we present the final calculation of the Born approximation for the Coulomb potential case which was given (as far as we know) only in a lesser known text by Landau and Lifshitz¹¹ (not to be confused with reference [1]). This derivation takes advantage of the fact that the Born approximation of the scattering amplitude, Eq. (3), is the Fourier transform of the potential $V(\vec{r})$ (apart from an overall constant). In our derivation below, we use an intermediate function $g(\vec{r})$ and initially assume it is sufficiently well-behaved so that the various steps in the calculation are permissible. An example of such a function is $g(\vec{r}) = \exp(-r^2/2)$. In the Supplementary Material,¹² we present a sufficient amount of the theory of generalized functions to enable us to make all the steps below mathematically rigorous.

The Fourier transform $\hat{g}(\vec{q})$ of the function $g(\vec{r})$ is given by,

$$\hat{g}(\vec{q}) = \int g(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} d^3r, \quad (19)$$

where in Cartesian coordinates (x, y, z) , we have, $d^3r = dx dy dz$, $d^3q = dq_x dq_y dq_z$, and all limits of integration are $-\infty$ to $+\infty$.

It is useful to recall the integration by parts formula for the case involving second derivatives. Suppose we have two sufficiently well-behaved functions $f(x)$ and $\varphi(x)$, then

$$\frac{d}{dx} (f' \varphi - f \varphi') = f'' \varphi - f \varphi''. \quad (20)$$

Integrating both sides of Eq. (20) over $x \in (-\infty, +\infty)$, we have

$$\int_{-\infty}^{\infty} f'' \varphi dx = [f' \varphi - f \varphi']_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f \varphi'' dx, \quad (21)$$

and remark that if the functions f and φ are such that the term in brackets on the right-hand side of Eq. (21) vanishes at the integration limits, then Eq. (21) reduces to

$$\int_{-\infty}^{\infty} f'' \varphi dx = \int_{-\infty}^{\infty} f \varphi'' dx. \quad (22)$$

If instead, f and φ were functions of (x, y, z) , then the integrals would be over the three-dimensional Euclidean space R^3 , and the term in brackets in Eq. (21) would contain two-dimensional integrals. Nevertheless, if f and φ and their derivatives vanish sufficiently fast at $\pm\infty$, we would obtain the generalization of Eq. (22), namely

$$\int_{R^3} \Delta f \varphi d^3r = \int_{R^3} f \Delta \varphi d^3r, \quad (23)$$

where Δ is the Laplacian. Using Eq. (23), one can show that the Fourier transform $\widehat{\Delta g}$ of $\Delta g = (\partial_x^2 + \partial_y^2 + \partial_z^2) g$, is

$$\int (\Delta g) e^{-i\vec{q} \cdot \vec{r}} d^3r = \int g \Delta (e^{-i\vec{q} \cdot \vec{r}}) d^3r = -q^2 \widehat{g}(\vec{q}), \quad (24)$$

where $q^2 = q_x^2 + q_y^2 + q_z^2$. Then from Eqs. (19) and (24), we have that

$$\widehat{g}(\vec{q}) = \int g(\vec{r}) e^{-i\vec{q} \cdot \vec{r}} d^3r = -\frac{1}{q^2} \int (\Delta g) e^{-i\vec{q} \cdot \vec{r}} d^3r. \quad (25)$$

In our case, using Eq. (3), we see that

$$f_B = -\frac{m}{2\pi} \int \frac{(\pm e^2)}{r} e^{-i\vec{q} \cdot \vec{r}} d^3r \quad (26)$$

$$= \mp \left(\frac{me^2}{2\pi} \right) \int \frac{e^{-i\vec{q} \cdot \vec{r}}}{r} d^3r := \mp \left(\frac{me^2}{2\pi} \right) \widehat{g}(\vec{q}), \quad (27)$$

and so

$$g(\vec{r}) = \frac{1}{|\vec{r}|}, \quad (28)$$

where $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$. At this point we use the equation,

$$\Delta \frac{1}{|\vec{r}|} = -4\pi\delta(\vec{r}). \quad (29)$$

Equation (29) is derived in the in the Supplementary Material¹² using the appropriate mathematical tools. Substituting Eq. (29) in Eq. (25), we find that

$$\hat{g}(\vec{q}) = \frac{4\pi}{q^2}, \quad (30)$$

which when substituted in Eq. (27) gives us the desired scattering amplitude

$$f_B = \mp \frac{2me^2}{q^2}. \quad (31)$$

It is clear that in the integration by parts which led to Eq. (24), we discarded several terms without any justification. Furthermore the resulting integrals in Eq. (24) do not exist for the $g(\vec{r})$ of Eq. (28). The preceding calculations and Eq. (29) can be made rigorous only in the context of the theory of generalized functions. In the Supplementary Material,¹² we present a concise introduction to the necessary formalism, derive Eq. (29), and prove Eq. (24).

V. SUMMARY

We saw that in Wentzel's approach in Sec. II, Eqs. (7)–(8), the prescription is: Perform the integration first and then take the limit $\lambda \rightarrow 0$. In Oppenheimer's approach in Sec. III, Eq. (16), the prescription is: Do the iterated integrals in the prescribed order. There is no escaping the fact that the Coulomb potential does not satisfy Eq. (6) and consequently evaluating the Born approximation requires special care. We hope that our simplification and clarification of Oppenheimer's calculation will be a useful addition to the usual textbook presentations on this subject. However we believe that Landau and Lifshitz derivation using the theory of generalized functions is the most satisfactory both from the mathematical and physics point of view since it does not require any extraneous assumptions nor an a priori knowledge of the final result.

VI. ACKNOWLEDGMENTS

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Appendix A: Units and dimensions

We use Planck natural units, $c = \hbar = 1$, and with respect to the electromagnetic equations, Gaussian units. Thus the Coulomb potential energy for two simple charges, each $\pm e$, $e > 0$, is

$$V(r) = \frac{\pm e^2}{r}, \quad (\text{A1})$$

where the signs, $(+, -)$, correspond to repulsion and attraction respectively. In our units the fine structure constant is dimensionless and

$$\frac{e^2}{\hbar c} \cong \frac{1}{137}. \quad (\text{A2})$$

From the Compton wavelength relation below,

$$\lambda_C = \frac{2\pi\hbar}{mc} = \frac{2\pi\hbar c}{mc^2}, \quad (\text{A3})$$

it follows that in natural units all quantities have dimensions of length, L , or inverse length. In particular $[E] = [p] = [m] = L^{-1}$.

It is also useful to keep in mind that since the differential cross section has dimensions of area,

$$\left[\frac{d\sigma}{d\Omega} \right] = L^2, \quad (\text{A4})$$

we have that the scattering amplitude has dimensions of length,

$$[f(p, \theta)] = L. \quad (\text{A5})$$

Appendix B: Review of scattering theory

In this Appendix we outline some relevant material leading to the Schrödinger equation for elastic scattering by a *central potential*,^{1,2,30,31} the definition of the scattering amplitude, $f(p, \theta)$, and the Born approximation, $f_B(p, \theta)$.

1. Reduction of the two-particle Hamiltonian

The Hamiltonian operator for two particles whose interaction is given by a function $V(|\vec{r}_1 - \vec{r}_2|)$ (which is invariant under the simultaneous translation of the \vec{r}_i 's) is

$$\mathbb{H} = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|), \quad (\text{B1})$$

where \vec{p}_1 and \vec{p}_2 are the momentum operators of particle 1 and 2 respectively. The corresponding Schrödinger equation is

$$\mathbb{H}\Psi(\vec{r}_1, \vec{r}_2) = E\Psi(\vec{r}_1, \vec{r}_2). \quad (\text{B2})$$

The canonical transformation,

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \quad (\text{B3})$$

$$\vec{p} = \frac{m_2\vec{p}_1 - m_1\vec{p}_2}{m_1 + m_2}, \quad \vec{P} = \vec{p}_1 + \vec{p}_2, \quad (\text{B4})$$

where using $r = |\vec{r}|$ as the distance between the two particles, and \vec{R} as the position of their center of mass, transforms the Hamiltonian of Eq. (B1) into

$$\mathbb{H} = \frac{\vec{P}^2}{2(m_1 + m_2)} + \left[\frac{\vec{p}^2}{2m} + V(r) \right], \quad (\text{B5})$$

$$= H_{cm} + H. \quad (\text{B6})$$

The mass m is the *reduced mass* of the system,

$$m = \frac{m_1 m_2}{m_1 + m_2}. \quad (\text{B7})$$

Thus the two-particle Hamiltonian of Eq. (B1) has been reduced to a sum of two *independent* Hamiltonians. H_{cm} is the Hamiltonian for the motion of the center of mass of the system, which moves as a free particle with constant momentum \vec{P} , while H is the Hamiltonian of a particle having mass m and moving in the potential energy field $V(r)$. It follows that the wavefunction Ψ of Eq. (B2) factors into two independent parts, and may be written as

$$\Psi(\vec{r}_1, \vec{r}_2) = e^{i\vec{P} \cdot \vec{R}} \psi(\vec{r}). \quad (\text{B8})$$

The Schrödinger equation for the wavefunction ψ is

$$\Delta\psi(\vec{r}) + 2m(E - V(r))\psi(\vec{r}) = 0, \quad (\text{B9})$$

where Δ is the Laplacian in spherical coordinates (r, θ, ϕ) . Since our potential depends only on r , the wavefunction ψ will be independent of the azimuthal angle ϕ . The energy $E = p^2/2m$.

Choosing the origin of our coordinate system so that $\vec{R} = 0$ and differentiating the second equation in Eqs. (B15) with respect to time, we obtain

$$\vec{p}_1 + \vec{p}_2 = 0, \quad (\text{B10})$$

while from the first equation of Eqs. (B4), we have that

$$\vec{p} = \vec{p}_1. \quad (\text{B11})$$

We let \vec{p}_1 , \vec{p}_2 be the incoming particle momenta in the center of mass system, and \vec{p}'_1 , \vec{p}'_2 the outgoing particle momenta. In elastic scattering the incoming and outgoing particles are the same, and, moreover, their kinetic energies must be equal before and after the collision; thus

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{(p'_1)^2}{2m_1} + \frac{(p'_2)^2}{2m_2}. \quad (\text{B12})$$

From Eqs. (B9)-(B11), it follows that

$$p = |\vec{p}_1| = |\vec{p}_2| = |\vec{p}'_1| = |\vec{p}'_2|. \quad (\text{B13})$$

Finally, we define the momentum transfer vector \vec{q} for later use as,

$$\vec{q} = \vec{p}'_1 - \vec{p}_1, \quad (\text{B14})$$

and using Eq. (B13), we have that

$$q^2 = 2p^2(1 - \cos \theta), \quad (\text{B15})$$

where θ is the the center of mass scattering angle, $\theta \in [0, \pi]$. In Fig. 1 below we show the ingoing and outgoing momenta for elastic scattering in the center of mass frame and the momentum transfer vector \vec{q} .

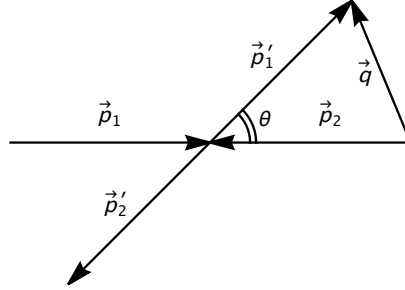


FIG. 1. We show the ingoing and outgoing momentum vectors for elastic scattering in the center of mass frame, and the momentum transfer vector \vec{q} .

2. The scattering amplitude

In the time-independent description of scattering, in the asymptotic region where $V(r) \simeq 0$, the total wavefunction consists of three parts. First, two plane waves, one representing the incident particles and another with the same momentum corresponding to the unscattered particles. These are indistinguishable in the time-independent description, and so we write (recall Eq. (B10))

$$\psi_{in} = e^{i\vec{p}\cdot\vec{r}} = e^{ipz}, \quad (\text{B16})$$

where we have chosen z to be along the incident beam direction (the polar axis). The third part of the wavefunction is an outgoing spherical wave corresponding to the scattered particles, which we write as,

$$\psi_{sc} \approx f(p, \theta) \frac{e^{ipr}}{r}. \quad (\text{B17})$$

We point out the following useful relations,

$$\vec{p}_1' = p \frac{\vec{r}}{r}, \quad (\text{B18})$$

$$\vec{p}_1' \cdot \vec{r} = p r, \quad (\text{B19})$$

$$\vec{p}_1 \cdot \vec{p}_1' = p^2 \cos \theta. \quad (\text{B20})$$

In fact then,

$$\psi(\vec{r}) = e^{ipz} + f(p, \theta) \frac{e^{ipr}}{r} + O\left(\frac{1}{r^2}\right). \quad (\text{B21})$$

The function $f(p, \theta)$ is the *scattering amplitude* and is in turn related to the *differential scattering section* by the relation

$$\frac{d\sigma}{d\Omega} = |f(p, \theta)|^2. \quad (\text{B22})$$

The differential cross section $d\sigma/d\Omega$ is equal to the *number of particles (of a given kind) scattered in the direction θ , per unit solid angle, per unit time, per unit incident flux, per scatterer*.

Although we have skipped a lot of important details, we would like to mention that the normalization of the asymptotic form of ψ in Eq. (B21) was chosen so as to agree with our definition of $d\sigma/d\Omega$. In summary, we see that in order to obtain $f(p, \theta)$, we have to solve the Schrödinger equation for ψ , and then find its asymptotic form, Eq. (B21).

We should add that the case of identical particles requires special care, since ψ has to be made symmetric or antisymmetric depending on whether the total spin of the incoming particles is even or odd respectively. We refer the interested reader to the best treatment on this subject, namely, section 137 of reference [1]. Here we give the result for simplest example: that of two, zero spin, identical bosons, in which case instead of Eq. (B22), we have

$$\frac{d\sigma}{d\Omega} = |f(p, \theta) + f(p, \pi - \theta)|^2. \quad (\text{B23})$$

3. The Born approximation

Schrödinger's equation (B9) can be converted into the integral equation,³¹

$$\psi(\vec{r}) = e^{ip\vec{r}} - \frac{m}{2\pi} \int \frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(r') \psi(\vec{r}') d^3r', \quad (\text{B24})$$

where $V(r')$ is again a centrally symmetric potential. The integral equation (B24) is exact and has the advantage that it is constructed so that the “boundary condition” of Eq. (B21) is satisfied automatically by the solution $\psi(\vec{r})$. It was shown by Born⁸ that the above integral equation may be solved by iterating ψ . The resulting series is called the Born series.

In the present paper we are concerned with the lowest order term in the Born series, the so-called *Born approximation*.¹ In order to calculate this term, we have to make some

reasonable approximations to Eq. (B24). If we assume that $V(r')$ decreases sufficiently rapidly as r' becomes large, then the domain of integration over r' is essentially finite and for very large r we can write

$$|\vec{r} - \vec{r}'| = r \left(1 + \frac{(r')^2}{r^2} - \frac{2\vec{r} \cdot \vec{r}'}{r^2} \right)^{1/2} \approx r - \frac{\vec{r} \cdot \vec{r}'}{r}. \quad (\text{B25})$$

Although we may further approximate $|\vec{r} - \vec{r}'| \sim r$ in the denominator of the integrand, the exponential depends sensitively on r' , no matter how large r is. Nonetheless, using Eq. (B18), we may write

$$e^{ip|\vec{r} - \vec{r}'|} \approx e^{ip\left(r - \frac{\vec{r} \cdot \vec{r}'}{r}\right)} = e^{ipr - i\vec{p}_1' \cdot \vec{r}'}. \quad (\text{B26})$$

Then, using Eqs. (B25) and (B26), we can simplify Eq. (B24)

$$\psi(\vec{r}) \approx e^{i\vec{p} \cdot \vec{r}} - \frac{m}{2\pi} \left(\frac{e^{ipr}}{r} \right) \int e^{-i\vec{p}_1' \cdot \vec{r}'} V(r') \psi(\vec{r}') d^3r', \quad (\text{B27})$$

and comparing with Eq. (B21), we obtain

$$f(p, \theta) = -\frac{m}{2\pi} \int e^{-i\vec{p}_1' \cdot \vec{r}'} V(r') \psi(\vec{r}') d^3r'. \quad (\text{B28})$$

It is important to realize that, although we did some approximations in obtaining the asymptotic expression of Eq. (B24), the expression for the scattering amplitude $f(p, \theta)$ above is *exact*. The lowest order approximation for $\psi(\vec{r}')$ is (see Eq. (B27)),

$$\psi(\vec{r}') \sim e^{i\vec{p}_1 \cdot \vec{r}'}, \quad (\text{B29})$$

thus the scattering amplitude Eq. (B28) becomes

$$f_B(p, \theta) = -\frac{m}{2\pi} \int e^{-i(\vec{p}_1' - \vec{p}_1) \cdot \vec{r}'} V(r') d^3r', \quad (\text{B30})$$

$$= -\frac{m}{2\pi} \int e^{-i\vec{q} \cdot \vec{r}'} V(r') d^3r', \quad (\text{B31})$$

where we have used the momentum transfer, \vec{q} , Eq. (B14). At this point we may drop the primes without any danger of confusion, and obtain our Eq. (3).

Appendix C: Oppenheimer's choice

Before proceeding, we remind the reader that the angle ϑ in the spherical coordinates below is not related to the scattering angle θ in Eq. (2). We now consider the momentum

transfer vector \vec{q} and the radial vector \vec{r} , with Cartesian components,

$$\vec{q} = (q^x, q^y, q^z), \quad (\text{C1})$$

$$\vec{r} = (x, y, z). \quad (\text{C2})$$

If we choose \vec{q} to be along the polar axis (i.e., the positive z -axis) then

$$\vec{q} = (0, 0, q). \quad (\text{C3})$$

The Cartesian components of \vec{r} expressed in spherical coordinates, (r, ϑ, ϕ) , are

$$\vec{r} = (r \sin \vartheta \cos \phi, r \sin \vartheta \sin \phi, r \cos \vartheta), \quad (\text{C4})$$

thus

$$\vec{q} \cdot \vec{r} = q r \cos \vartheta. \quad (\text{C5})$$

Likewise if, instead, we express the Cartesian components of \vec{r} in cylindrical coordinates (z, ϱ, φ) , we have

$$\vec{r} = (\varrho \cos \varphi, \varrho \sin \varphi, z), \quad (\text{C6})$$

and thus we have

$$\vec{q} \cdot \vec{r} = q z. \quad (\text{C7})$$

Oppenheimer¹⁰ chose \vec{p}_1 to be along the positive z -axis, so that

$$\vec{p}_1 = (0, 0, p). \quad (\text{C8})$$

Then without loss of generality \vec{p}'_1 , may be written as (see Fig. 1)

$$\vec{p}'_1 = (p \sin \theta, 0, p \cos \theta), \quad (\text{C9})$$

hence

$$\vec{q} = (p \sin \theta, 0, p \cos \theta - p). \quad (\text{C10})$$

So using Eq. (C6) we obtain

$$\vec{q} \cdot \vec{r} = p \varrho \sin \theta \cos \varphi + p(\cos \theta - 1)z. \quad (\text{C11})$$

As the reader may check by consulting reference [10], the above choice complicates his calculation considerably.

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